Definite Integrals

Let $f(x)$ be a continuous function defined on $[a, b]$ and let $F(x)$ be the anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

This is called the Fundamental Theorem of Integral Calculus.

* $\int_1^2 x\,dx$.

Sol: $\int_1^2 x\,dx = \dfrac{x^{1+1}}{1+1} + c\Big|_1^2 = \dfrac{x^2}{2} + c\Big|_1^2$

$$= \frac{2^2}{2} - \frac{1^2}{2} = \frac{4}{2} - \frac{1}{2} = 3/2$$

* $\int_0^2 e^x\,dx$

Sol: $\int_0^2 e^x\,dx = e^x\Big|_0^2 = e^2 - e^0 = e^2 - 1$

* $\int_1^4 (x+2)\,dx$

Sol: $\int_1^4 x\,dx + \int_1^4 2\,dx$

$$= \left.\frac{x^2}{2}\right|_1^4 + \left.2x\right|_1^4$$

$$= \frac{1}{2}\left[4^2 - 1^2\right] + 2\left[4-1\right]$$

$$= \frac{1}{2} \times 15 + 6 = \frac{15}{2} + 6 = \underline{\underline{27/2}}$$

Angle in Degree	Angle in Radian
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
360°	2π

eg: $\sin 30^\circ = \sin(\pi/6) = 1/2$

$\tan 45^\circ = \tan(\pi/4) = 1.$

Evaluation of definite Integral by Substitution

* $\int_0^1 \frac{2x}{1+x^2}\,dx$

* Let $u = 1+x^2$, when $x = 0$, $u = 1$
$x = 1$, $u = 2$

$\frac{du}{dx} = \frac{d(1+x^2)}{dx}$

$\Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\therefore \int_1^2 \frac{\cancel{2x}}{u} \cdot \frac{du}{\cancel{2x}} = \int_1^2 \frac{1}{u}\,du$

$= \log u \Big|_1^2$

$= \log 2 - \log 1$

$= \log 2$

* $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

Sol: Let $u = \tan^{-1}x$

when $x = 0$, $u = \tan^{-1}(0) = 0$

when $x = 1$, $u = \tan^{-1}(1) = \pi/4$

$$\frac{du}{dx} = \frac{d}{dx}\tan^{-1}x$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \Rightarrow dx = du\,(1+x^2)$$

$$\therefore \int_0^{\pi/4} \frac{u}{\cancel{1+x^2}} \cdot du\,(\cancel{1+x^2}) = \int_0^{\pi/4} u\,du$$

$$= \left.\frac{u^2}{2}\right|_0^{\pi/4}$$

$$= \frac{1}{2} \cdot \left[\left(\frac{\pi}{4}\right)^2 - \frac{0^2}{2} \right]$$

$$= \frac{1}{2} \cdot \frac{\pi^2}{16}$$

$$= \underline{\underline{\pi^2/32}}$$

$$* \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$$

Sol: Let $u = \cos x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \pi/2$, $u = \cos \pi/2 = 0$

Now,

$$\frac{du}{dx} = \frac{d\cos x}{dx} \Rightarrow \frac{du}{dx} = -\sin x$$

$$\Rightarrow dx = \frac{du}{-\sin x}$$

$$\therefore \int_1^0 \frac{\cancel{\sin x}}{1+u^2} \cdot \frac{du}{-\cancel{\sin x}} = -\int_1^0 \frac{1}{1+u^2}\, du$$

$$= - \tan^{-1} u \Big|_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \pi/4\right]$$

$$= \pi/4$$

Properties of Definite Integrals

1) Integration is independent of change of variables

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

eg: $\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{0}{2} = \underline{1/2}$

$\int_0^1 y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{0}{2} = \underline{1/2}$

2) If the limits of a definite Integral are interchanged, then its value is changed by sign.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

eg: $$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{0}{2} = \underline{\underline{1/2}}$$

$$\int_1^0 x\,dx = \frac{x^2}{2}\Big|_1^0 = \frac{0}{2} - \frac{1}{2} = \underline{\underline{-1/2}}$$

3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where c is any number

eg: $$\int_1^4 f(x)\,dx, \quad f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 \le x \le 4 \end{cases}$$

$$\therefore \int_1^4 f(x)\,dx = \int_1^2 (2x+8)\,dx + \int_2^4 6x\,dx$$

$$= \cancel{2}\cdot\frac{x^2}{\cancel{2}}\Big|_1^2 + 8\cdot x\Big|_1^2 + \overset{3}{\cancel{6}}\cdot\frac{x^2}{\cancel{2}}\Big|_2^4$$

$$= x^2\Big|_1^2 + 8\cdot[2-1] + 3\cdot x^2\Big|_2^4$$

$$= [4-1] + 8 + 3\cdot[16-4]$$

$$= 3 + 8 + 36 = \underline{47}$$

4) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

eg: LHS $= \int_1^2 x\,dx = \left.\frac{x^2}{2}\right|_1^2 = \frac{1}{2}[4-1] = \underline{\underline{3/2}}$

RHS $= \int_1^2 (2+1-x)\,dx$

$= \int_1^2 (3-x)\,dx$

$= 3\left.x\right|_1^2 - \left.\frac{x^2}{2}\right|_1^2$

$= 3.[2-1] - \frac{1}{2}.[4-1]$

$= 3 - \frac{3}{2} = \underline{\underline{3/2}}$

$\therefore$ LHS = RHS

5) $\int_{-a}^{a} f(x)\,dx = \begin{cases} = 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ = 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$

Even function : if $f(-x) = f(x)$

eg: $f(x) = x^2$

$f(-x) = (-x)^2 = \underline{\underline{x^2}}$

odd function: if $f(-x) = -f(x)$

eg: $f(x) = x^3$

$f(-x) = (-x)^3 = \underline{\underline{-x^3}}$

* $\int_{-\pi/2}^{\pi/2} \cos x \, dx$

Sol: Since $\cos(-x) = \cos x$, it is an even function.

$\therefore \int_{-\pi/2}^{\pi/2} \cos x \, dx = 2\int_{0}^{\pi/2} \cos x \, dx$

$= 2 \left[\sin x\right]_0^{\pi/2}$

$= 2 \left[1-0\right] = \underline{\underline{2}}$.